On "Non-Geometric" Contribution To The Entropy Of Black Hole Due To Quantum Corrections.

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Abstract

The quantum corrections to the entropy of charged black holes are calculated. The Reissner-Nordstrem and dilaton black holes are considered. The appearance of logarithmically divergent terms not proportional to the horizon area is demonstrated. It is shown that the complete entropy which is sum of classical Bekenstein-Hawking entropy and the quantum correction is proportional to the area of quantum-corrected horizon.

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The classical Bekenstein-Hawking entropy of four-dimensional black hole is known to be proportional to the area of horizon:

$$S_{BH} = \frac{1}{4} \frac{A_h}{\kappa},$$ (1)

where κ is gravitational constant [1]. Roughly speaking, the horizon is two-dimensional surface which separates the whole space on two different regions the free exchange of information between which is impossible. Thus, the outside observer does not have information about states of quantum field in the region inside the horizon and therefore must trace over all such states. The entropy, characterizing this unknowledge, turns out to be determined only by geometry of the surface separating these two regions, namely it is proportional to the area of the surface. This fact occurs to be not feature of only gravitational objects but rather typical [2].

It is reasonable to ask whether this geometric character of the black hole entropy remains valid when the quantum corrections (say, due to quantum fluctuations of matter fields in the black hole background) are taken into account.

Approximating the metric of black hole of infinitely large mass by more simple Rindler metric, it was shown [3] that quantum correction to (1) again takes the geometric character:

$$S^q = \frac{1}{48\pi} \frac{A_h}{\epsilon^2}$$ (2)

though it is divergent when the ultraviolet cut-off ϵ tends to zero. This divergence was related with information loss in the black hole [4].

However, recently [5] we shown that the Rindler metric is not good model for the black hole space-time. The reason is that the horizon surfaces of black hole (sphere) and Rindler space (plane) are topologically different. For black hole of finite mass at the same time with (2) one also observes the logarithmically divergent mass independent term:

$$S^q = \frac{1}{48\pi} \frac{A_h}{\epsilon^2} + \frac{1}{45} \log \frac{\Lambda}{\epsilon},$$ (3)

where Λ is infra-red cut-off. This term does not have the geometric character and resembles the quantum correction to entropy of two-dimensional black hole [6]. Generally, entropy is defined up to arbitrary additive constant. Hence, one could assume that this

term is not essential and does not influence the physics. In this note we show (on example of charged black holes) that appearance of such, non-geometric, logarithmically divergent terms is typical in four dimensions. In general case, these terms depend on the characteristics of the black hole (charge, mass, etc.) and therefore can not be neglected as non-essential additive constants. We use the path integral method of Gibbons and Hawking [7] to calculate the corrections to entropy of black hole. The basic formulas can be found in [5].

In the Euclidean path integral approach to statistical field system taken under temperature $T = (2\pi\beta)^{-1}$ one considers the fields which are periodical with respect to imaginary time τ with period $2\pi\beta$. For arbitrary β the classical black hole metric is known to have conical singularity which disappears only for special Hawking inverse temperature β_H.

Let matter is described by the action:

$$I_{mat} = \frac{1}{2} \int (\nabla\Phi)^2 \sqrt{g} d^4 x \tag{4}$$

Then contribution to the energy and entropy due to matter fluctuations is given by

$$E^q = \frac{1}{2\pi} \partial_\beta I_{eff}(\beta, \Delta)|_{\beta=\beta_H} \quad S^q = (\beta\partial_\beta - 1)I_{eff}(\beta, \Delta)|_{\beta=\beta_H}, \tag{5}$$

where $\Delta = \nabla_\mu \nabla^\mu$ is the Laplace operator; $I_{eff}(\beta, \Delta) = \frac{1}{2} \ln det\Delta_{g_\beta}$ is the one-loop effective action calculated in the classical black hole background with conical singularity at the horizon. In order to take derivative ∂_β in (5) we assume that β is slightly different of β_H.

The logarithm of determinant in the De Witt-Schwinger proper time representation is as follows:

$$\log det\Delta = -\int_{\epsilon^2}^{\infty} s^{-1} Tr(e^{-s\Delta}), \tag{6}$$

where the integral over s is cutted on the lower limit under $\epsilon^2 = L^{-2}$, L is maximal impulse.

In four dimensions we have the asymptotic expansion:

$$Tr(e^{-s\Delta}) = \frac{1}{(4\pi s)^2} \sum_{n=0}^{\infty} a_n s^n \tag{7}$$

The divergent part of the effective action is given by

$$I_{eff} = -\frac{1}{32\pi^2} (\frac{1}{2} a_0 \epsilon^{-4} + a_1 \epsilon^{-2} + a_2 \log(\frac{\Lambda}{\epsilon})^2) \tag{8}$$

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The black hole metric in vicinity of horizon ($\rho = 0$) has the form:

$$ds^2 = \alpha^2(\rho^2 + C\rho^4)d\phi^2 + d\rho^2 + (\gamma_{ij}(\theta) + h_{ij}(\theta)\rho^2)d\theta^i d\theta^j \tag{9}$$

where $C = const$, $\alpha = \frac{\beta}{\beta_H}$ and we introduced new coordinate $\phi = \beta^{-1}\tau$ which has period 2π. Near the horizon ($\rho = 0$) this Euclidian space looks as direct product $M_\alpha = C_\alpha \otimes \Sigma$. C_α is two dimensional cone with metric $ds^2 = \alpha^2\rho^2 d\phi^2 + d\rho^2$, Σ is the horizon surface with metric $\gamma_{ij}(\theta)$. It was shown recently [8] that for background like this the coefficients in the expression (7) take the form

$$a_n = a_n^{reg} + a_{\alpha,n} \tag{10}$$

where a_n^{reg} are standard coefficients $a_n = \int_{M_\alpha} a_n(x,x)d\Omega(x)$, given by the integrals over the smooth domain of M_α; the coefficients $a_{\alpha,n}$ are surface terms determined by integrals over the horizon Σ: [1]

$$a_{\alpha,0} = 0; \quad a_{\alpha,1} = \frac{\pi}{3}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma \sqrt{\gamma}d^2\theta \ ;$$
$$a_{\alpha,2} = \frac{\pi}{18}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma R\sqrt{\gamma}d^2\theta$$
$$-\frac{\pi}{180}\frac{(1-\alpha)(1+\alpha)(1+\alpha^2)}{\alpha^3}\int_\Sigma (R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta)\sqrt{\gamma}d^2\theta \tag{11}$$

where n^i are two vectors orthogonal to surface Σ ($n_i^\mu n_j^\nu g_{\mu\nu} = \delta_{ij}$). For metric (9) we may take $n_1^\mu = ((\alpha\rho)^{-1}, 0, 0, 0)$, $n_2^\mu = (0, 1, 0, 0)$.

For metric (9) we obtain at $\rho = 0$:

$$R = R_\Sigma - 6C - 4\gamma^{ij}h_{ij}$$
$$R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta = 6C - 2\gamma^{ij}h_{ij}, \tag{12}$$

where R_Σ is scalar curvature determined with respect to two-dimensional metric γ_{ij}.

Inserting (10), (11), (12) into (5) we obtain for correction to the entropy

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + (\frac{1}{18} - \frac{1}{16\pi}\int_\Sigma(\frac{4}{5}C + \frac{2}{5}\gamma^{ij}h_{ij})\sqrt{\gamma}d^2\theta)\log\frac{\Lambda}{\epsilon} \tag{13}$$

where we used the fact that the horizon surface Σ is sphere and hence $\frac{1}{4\pi}\int_\Sigma R_\Sigma\sqrt{\gamma}d^2\theta = 2$; $A_\Sigma = \int_\Sigma \sqrt{\gamma}d^2\theta$ is the horizon area.

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[1] Our convention for the curvature and Ricci tensor is $R^\alpha_{\ \beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\ \nu\beta} - ...$, and $R_{\mu\nu} = R^\alpha_{\ \mu\alpha\nu}$.

If we start with black hole metric written in the Schwarzschild like form

$$ds^2 = \beta^2 g(r)d\phi^2 + \frac{1}{g(r)}dr^2 + r^2\tilde{g}_{ij}(\theta)d\theta^i d\theta^j \tag{14}$$

where $\tilde{g}_{ij}(\theta)$ is metric of 2D sphere, then introducing new radial coordinate $\rho = \int g^{-1/2}dr$ we obtain in vicinity of horizon (which is determined as simple zero of $g(r)$) the metric in the form (9) where $\gamma_{ij} = r_h^2\tilde{g}_{ij}$, $h_{ij} = \frac{r_h}{\beta_H}\tilde{g}_{ij}$ and $C = \frac{1}{6}g_r''|_{r_h}$; r_h is radius of the horizon sphere. The corresponding Hawking temperature is $\beta_H = 2(g_r'(r_h))^{-1}$.

Finally we get for the quantum correction to the entropy:

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(\frac{1}{18} - \frac{A_\Sigma}{20\pi}\left(\frac{1}{6}g_r''|_{r_h} + \frac{1}{r_h\beta_H}\right)\right)\log\frac{\Lambda}{\epsilon} \tag{15}$$

We see that logarithmic term in (15) is formally proportional to the horizon area A_Σ. However, the coefficient of proportionality depends on the background black hole geometry and therefore the whole expression does not take the form (1).

Let us consider some particular examples.

Example 1. Reissner-Nordstrem black hole.

The charged black hole is described by metric (14) with

$$g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad M \geq Q. \tag{16}$$

The largest horizon is located at

$$r_h = M + \sqrt{M^2 - Q^2} \tag{17}$$

The corresponding Hawking inverse temperature is

$$\beta_H = \frac{r_h^2}{\sqrt{M^2 - Q^2}} \tag{18}$$

and for the second derivative g'' we have

$$g''|_{r_h} = 2r_h^{-4}(3Q^2 - 2M^2 - 2M\sqrt{M^2 - Q^2}) \tag{19}$$

Inserting (17)-(19) into (15) we obtain for the quantum correction to the entropy:

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(\frac{1}{18} - \frac{M}{15r_h}\right)\log\frac{\Lambda}{\epsilon} \tag{20}$$

When mass M becomes infinitely large, $r_h \to 2M$ and (20) coincides with (3). It is interesting to note that for extreme black hole ($M = Q$, $\beta_H = \infty$, $r_h = M$) the second term in (20) becomes negative:

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{90}\log\frac{\Lambda}{\epsilon} \tag{21}$$

However, this does not mean that the whole expression (21) is negative since in limit $\epsilon \to 0$ the first positive term is dominant.

Example 2. Dilaton charged black hole.

The metric of dilaton black hole having electric charge Q and magnetic charge P takes the form [9]:

$$ds^2 = gdt^2 + g^{-1}dr^2 + R^2d\Omega \tag{22}$$

with metric function

$$g(r) = \frac{(r - r_+)(r - r_-)}{R^2}, \quad R^2 = r^2 - D^2 \tag{23}$$

where D is the dilaton charge: $D = \frac{P^2 - Q^2}{2M}$. The outer and the inner horizons are defined as follows

$$r_\pm = M \pm r_0 ; \quad r_0^2 = M^2 + D^2 - P^2 - Q^2 \tag{24}$$

Near the outer horizon we have

$$R^2 = R_+^2 + \frac{r_+}{\beta_h}\rho^2, \quad R_+^2 = r_+^2 - D^2$$

and the metric (22) takes the form (9). The Hawking temperature β_H is

$$\beta_H = \frac{2(r_+^2 - D^2)}{(r_+ - r_-)} \tag{25}$$

and for the second derivative g_r'' we have

$$g_r''(r_h) = \frac{2}{(r_+^2 - D^2)^2}(r_+^2 - D^2 - 2(r_+ - r_-)r_+) \tag{26}$$

From general expression (15) we get for this type of black hole

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(-\frac{1}{90} + \frac{2}{15}\frac{r_+(r_+ - r_-)}{(r_+^2 - D^2)} + \frac{1}{10}\frac{(r_+ - r_-)}{r_+}\right)\log\frac{\Lambda}{\epsilon} \tag{27}$$

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where $A_\Sigma = 4\pi(r_+^2 - D^2)$.

It is instructive to consider the black hole with only electric charge ($P = 0$). Then $r_0 = M - \frac{Q^2}{2M}$ ($2M^2 > Q^2$), $\frac{r_+(r_+-r_-)}{(r_+^2-D^2)} = 1 - \frac{Q^2}{4M^2}$ and expression (27) takes the form

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(\frac{1}{18} + \frac{1}{15}\left(\frac{(2M^2 - Q^2)}{2M^2} - \frac{1}{5}\left(\frac{2M^2 - Q^2}{4M^2 - Q^2}\right)\right)\right)\log\frac{\Lambda}{\epsilon} \tag{28}$$

For large M we again obtain result (3).

In the case of dilaton extreme black hole, $2M^2 = Q^2$, the horizon area vanishes, $A_\Sigma = 0$, and the whole black hole entropy is determined only by the logarithmically divergent term

$$S^q_{extr} = \frac{1}{18}\log\frac{\Lambda}{\epsilon} \tag{29}$$

Notice that (29) is positive. Expression (29) is very similar to the entropy of two-dimensional black hole [6]. This can be considered as an additional justifying the point that the dilaton extreme black hole is effectively two-dimensional that was widely exploted recently [10].

Thus, we demonstrated on number of examples the appearance of logarithmically divergent terms in quantum correction to the entropy which are not proportional to the horizon area. One could conclude from this that the classical law (1) is broken due to the quantum corrections. However, one can show that the complete black hole entropy

$$S = S_{BH} + S^q \tag{30}$$

which is sum of classical Bekenstein-Hawking entropy (1) and the quantum correction again takes the form (1) being defined with respect to the renormalized quantities. The renormalized gravitational constant κ_{ren} is determined as follows [5]:

$$\frac{1}{\kappa_{ren}} = \frac{1}{\kappa} + \frac{1}{12\pi\epsilon^2} \tag{31}$$

Then (30) can be written in the form similar to (1):

$$S = \frac{1}{4\kappa_{ren}}A_{\Sigma,\,ren} \tag{32}$$

if we define the quantum corrected radius of horizon $r_{h,\,ren}$ as follows

$$4\pi r_{h,\,ren}^2 = 4\pi r_h^2 + \eta l_{pl}^2 \tag{33}$$

where $l_{pl}^2 = \kappa_{ren}$ is the Planck length; quantity $\eta = \eta(M, Q) \log \frac{\Lambda}{\epsilon}$ absorbs the logarithmic divergence of (30) and in general depends on bare black hole characteristics: M, Q, etc. For the Schwarzschild black hole η is positive constant. The expression like (33) appears in the work of York [11] describing the quantum fluctuations of the horizon and recently in [12] as result of quantum deformation of the Schwarzschild solution. On the other hand, for the charged Reissner-Nordstrem black hole we have $\eta = (\frac{2}{9} - \frac{4M}{15r_h}) \log \frac{\Lambda}{\epsilon}$ and for the extreme black hole $(Q = M)$ η is negative.

Expression (33) means that quantum corrections result in the shifting the horizon radius by the Planck distance. For the charged black hole with $Q < M < \sqrt{\frac{25}{24}}Q$ the quantum corrections decrease the horizon radius while for $M > \sqrt{\frac{25}{24}}Q$ it is increasing. The quantum corrected entropy is determined then with respect to this quantum corrected horizon in such a way that the law (1) remains valid. For massive black hole $(M >> M_{pl})$ this shifting of horizon is negligible. However, it becomes essential and important for black hole of the Planck mass.

One of the reasons for (33) to be hold could be the renormalization of mass of the black hole that can be calculated in principle from (5). One must take into account the boundary terms in the effective action which contribute to the energy (in (8) we neglected such a boundary terms). On the other hand, (33) can be considered as a result of deformation on small distances of the Schwarzschild solution due to quantum corrections (see [12]).

Concluding, we calculated the quantum correction to the entropy of charged black holes and demonstrated the appearance of logarithmically divergent terms not proportional to the horizon area. Nevertheless, we shown that the complete entropy which is sum of classical Bekenstein-Hawking entropy and the quantum correction is proportional to the area of the renormalized horizon.

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